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Dynamics of coupled topological solitons in a weakly coupled discrete sine–Gordon system with local impurities

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Abstract. We study the dynamics of kinks and/or antikinks in two weakly coupled discrete sine–Gordon chains with local impurities. We use the Lagrangian formalism to derive the effective equations of motion for the soliton coordinates, which takes into account the inhomogeneities, the discreteness of the lattice and the coupling between chains. It appears that the repulsive or the attractive character of the impurities depends on the soliton polarities and on competition between the impurities related to the elastic constants, the substrate potential barriers and the coupling between the chains. Conditions for the reflection and the attraction of low-velocity kink–kink, kink–antikink and antikink–antikink pairs are obtained and the threshold velocities for soliton reflection by impurities are derived.

1. Introduction

The study of interactions of linear and non-linear waves with impurities is a subject of considerable interest in various branches of physics. Non-linear and non-integrable equations are always obtained from disordered and inhomogeneous systems. The Φ^4 and sine–Gordon equations, which model the propagation of waves in Frenkel–Kontorova lattices, in Josephson transmission lines, in charge-density-wave systems and in ferroelectric and ferromagnetic materials, are widely studied examples.

In general, during their interaction with impurities, the topological solitons (kinks, fluxons, charge density waves and domain walls) may be (i) captured and (ii) reflected or transmitted with more or less distortion of their structure and a drastic change in their dynamical behaviour (Kivshar and Malomed 1989, Woafo and Kofané 1992). The generation of new degrees of freedom (e.g. impurity modes) has also been obtained (Fraggis *et al* 1989, Braun and Kivshar 1991). As a consequence, the transport properties (transmission coefficient, diffusion constant and conduction) of linear and non-linear one-dimensional systems, which are sensitive to the presence of imperfections and disorder, are therefore modified. In particular, in the presence of local impurities, the diffusion constant of kinks depends on the total character of the kink–impurity interaction (see Braun and Kivshar 1991 and references therein).

Owing to a lack of mathematical formalism, the study of the effects of inhomogeneities and disorder in physical models has so far been limited in the continuum limit. However, in the last 15 years, some progress has been made in view of analysing the discreteness effects in systems where the kink width is comparable to the lattice constant. It has been shown that the translational invariance of the kink motion in the continuum lattice is destroyed by radiative damping and the lattice spacing periodical or many-periodical

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variation of kink parameters: a kink of a discrete lattice has a periodically varying mass and moves in a periodic Peierls–Nabarro (PN) potential (Peyrard and Remoissenet 1982, Combs and Yip 1983, Peyrard and Kruskal 1984, Boesch *et al* 1989, Woafo *et al* 1991a, 1992a and references therein). This evidently leads to a renormalization of the kink diffusion constant (Combs and Yip 1984, Kunz and Combs 1985) and other thermodynamic properties (Trullinger and Sasaki 1987, Willis and Boesch 1990, Woafo *et al* 1992b).

In non-Hamiltonian and inhomogeneous lattices, the discreteness effects give rise to other phenomena. For instance, Peyrard and Kruskal (1984) showed after simulation that the velocity of a discrete kink driven by a small external force evolves by steps. That is, for a large range of applied forces, the final velocity remains almost constant and then jumps to another value where it is again constant for a new range of the force. Those steps occur at some critical velocities for which the emission of phonons, due to lattice discreteness, is absent. Recently, it has been shown that in the presence of a small external force and small damping, the discrete kink moves like a damped driven particle in the PN potential (Pouget *et al* 1989). When the constant external force is applied in the Φ^4 model, it destroys the symmetry of the substrate potential and the asymmetric kink moves in a PN potential whose barrier decreases when the force increases (Woafo *et al* 1991b). In recent papers, the problem of kink interactions with local impurities in a sine–Gordon (SG) lattice has been analysed. Considering the continuum limit, it has been demonstrated that, if the impurity mass is large enough in comparison with the standard mass in the SG model, a higher-velocity kink will be reflected while a lower-velocity kink will pass. However, if the impurity mass is not too large, the kink can pass the impurity almost freely at any initial velocity (Zhang Fei *et al* 1992). But, this is not the same in the discrete limit, where it is seen that the small impurity mass also leads to a reflection (Braun and Kivshar 1990, 1991).

Our aim in this paper is to study the influence of various types of impurity (mass impurities, elastic spring impurities, substrate potential barrier impurities and interchain coupling impurities) on the motion of toplogical solitons in two weakly coupled SG chains. Interesting results have already been obtained for the homogeneous coupled model (Woafo *et al* 1992a).

The organization of the paper is as follows. In section 2, we present the discrete model and the various types of impurity. Section 3 deals with the derivation of the equations of motion, which are analysed in section 4. In section 5, we conclude our work by outlining some interesting problems that are still under investigation.

2. Weakly coupled sine–Gordon systems with local impurities

We consider a weakly coupled discrete SG system, which can be derived from the following Lagrangian:

$$
\begin{aligned}
\mathrm{L} = \sum \Bigg\{ & \tfrac{1}{2} m_i \left(\frac{\mathrm{d}\phi_{1,i}}{\mathrm{d}t}\right)^2 + \tfrac{1}{2} M_i \left(\frac{\mathrm{d}\phi_{2,i}}{\mathrm{d}t}\right)^2 - \tfrac{1}{2} k_i (\phi_{1,i+1} - \phi_{1,i})^2 - \tfrac{1}{2} g_i (\phi_{2,i+1} - \phi_{2,i})^2 \\
& - \tfrac{1}{2}\epsilon_i \left[1 - \cos\left(\frac{2\pi}{a}\right)\phi_{1,i}\right] - \tfrac{1}{2}\gamma_i \left[1 - \cos\left(\frac{2\pi}{a}\right)\phi_{2,i}\right] \\
& + \beta_i (\phi_{1,i+1} - \phi_{1,i})(\phi_{2,i+1} - \phi_{2,i}) + \alpha_i \left[1 - \cos\left(\frac{2\pi}{a}\right)(\phi_{1,i} - \phi_{2,i})\right] \Bigg\}.
\end{aligned}
\tag{2.1}
$$

We have denoted by $\phi_{j,i}$ (with $j = 1, 2$) the displacement of the particle in cell i of the chain j. The constant a is the period of the substrate potential. The first two terms in (2.1) are the kinetic energies of the two chains (t being the time variable). The following four terms describe the potential energy, which consists of elastic energy (the two square terms) and the energy of interaction of a particle with the substrate (the two cosine terms). The last two terms denote the energy of interaction between the chains. They take into account the interaction between the relative displacements of particles in both chains and the interaction between a particle of one chain with its nearest neighbour of the other chain (Coutinho *et al* 1981, Braun *et al* 1988). For the sake of generality, we have assumed that the masses (m_i, M_i), the elastic constants (k_i, g_i) and the substrate potential barriers (ϵ_i, γ_i) in the site i of the lattice are different from one chain to another, but are related by equation (2.3) below. The coupling coefficients β_i and α_i are considered as small quantities ($|\beta_i|$ and $|\alpha_i| \ll 1$) throughout this paper.

We introduce the impurities in the discrete model by assuming that the masses, the spring or elastic constants, the substrate potential barriers and the coupling coefficients are present as single-point impurities on lattice cells l, p and n. That is

$$m_i = m\left(1 + \frac{\Delta m}{m}\delta_{il}\right) \qquad k_i = k\left(1 + \frac{\Delta k}{k}\delta_{il}\right) \qquad \epsilon_i = \epsilon\left(1 + \frac{\Delta \epsilon}{\epsilon}\delta_{il}\right) \tag{2.2a}$$

$$M_i = M\left(1 + \frac{\Delta M}{M}\delta_{ip}\right) \qquad g_i = g\left(1 + \frac{\Delta g}{g}\delta_{ip}\right) \qquad \gamma_i = \gamma\left(1 + \frac{\Delta \gamma}{\gamma}\delta_{ip}\right) \tag{2.2b}$$

$$\beta_i = \beta\left(1 + \frac{\Delta \beta}{\beta}\delta_{in}\right) \qquad \alpha_i = \alpha\left(1 + \frac{\Delta \alpha}{\alpha}\delta_{in}\right). \tag{2.2c}$$

The impurities on the first chain occur on cell l, those of the second chain on cell p and the one associated with interchain coupling on cell n. Taking l, p and n as different integers has the advantage of considering the general situation that may arise in coupled chains.

In equations (2.2) Δm, Δk, $\Delta\epsilon$, ΔM, Δg, $\Delta\gamma$, $\Delta\beta$ and $\Delta\alpha$ stand for small variations of the corresponding parameters on point impurities. They may be positive or negative. We assumed for simplicity that the constants m, M, k, g, ϵ and γ are related by the equation

$$M/m = g/k = \gamma/\epsilon. \tag{2.3}$$

For the homogeneous system, the Lagrangian (2.1) leads, in the continuum limit, to two coupled and non-linear partial differential equations, namely the coupled SG equations. When $\beta = \alpha = 0$, the system admits large-amplitude soliton solutions (topological solitons or kinks), which, with the parameters of the first chain, have the form

$$\phi_1(x, t) = \frac{2a}{\pi}\tan^{-1}\left(\sigma_1 \frac{\omega_0(x - Vt)}{C_0(1 - V^2/C_0^2)^{1/2}}\right) \tag{2.4a}$$

where x is the space variable and

$$\omega_0^2 = 2\pi^2\epsilon/ma^2 \qquad C_0^2 = kb^2/m. \tag{2.4b}$$

In equation (2.4a), V represents the soliton velocity and $\sigma_1 = \pm 1$ is the soliton polarity ($\sigma_1 = 1$ for kink and $\sigma_1 = -1$ for antikink).

When β and α differ from zero, the linear and trivial non-linear excitations ($\phi_1 = 0$ and $\phi_2 \neq 0$, $\phi_1 \neq 0$ and $\phi_2 = 0$, $\phi_2 = -\phi_1$) can easily be obtained. In the general case where there are two different non-linear excitations in both chains, it has been shown by means of the classical perturbation method (β and α being the perturbation parameters) that the interaction between the chains distorts the soliton shape (Kivshar and Malomed 1988, Zhang 1987, Woafo *et al* 1992a, Dikandé and Kofané 1992). However, numerical calculation reveals that the distortions have a slight influence on the soliton energy. Therefore, one can admit the propagation, under the action of the coupling terms, of undistorted solitons (e.g. the form (2.4)). In the framework of the McLaughlin and Scott (1978) perturbation theory for non-linear waves, this problem has been considered by Braun *et al* (1988). It has been shown that the interaction (attraction or repulsion) between kinks belonging to different chains is determined by the soliton relative polarity σ and the interchain coupling (e.g. by the parameters α, β and $\sigma = \sigma_1\sigma_2$).

The effect of σ, α and β has also been analysed in the discrete limit. The collective coordinate method associated with Dirac's formalism for constrained Hamiltonian dynamics has been used to derive the equations of motion for centres of coupled topological solitons (Woafo *et al* 1992a). It appears that in the case $\beta > 0$ (with $\alpha = 0$) the coupling reduces (increases) the trapping processes when the solitons have the same polarities (different polarities). The inverse situation is observed when $\beta < 0$. In section 3, we consider the soliton motion in the inhomogeneous and discrete system. The Lagrangian formalism is used to derive the effective equations of motion of the coupled solitons.

3. Equations of motion for coupled solitons in the inhomogeneous system

To analyse the dynamics of topological solitons in the coupled inhomogeneous model, let us introduce the dimensionless variables for the displacement fields $\phi_{j,i}$ and the time t respectively:

$$y_{j,i} = (2\pi/a)\phi_{j,i} \qquad \text{and} \qquad \tau = (C/a)t \tag{3.1}$$

with

$$C^2 = a^2k/m = a^2g/M.$$

Assuming that the interchain coupling and the impurities act only on the soliton positions in the lattice, the dimensionless variables $y_{j,i}$ can be approximated, in the non-relativistic regime, by the *ansatz* (Pouget *et al* 1989, Braun and Kivshar 1991)

$$y_{j,i} = 4\tan^{-1}[\exp(\theta_j\xi_{j,i})] \tag{3.2a}$$

where

$$\xi_{j,i} = i - X_j \tag{3.2b}$$

and

$$\theta_j = \sigma_j\mu \tag{3.2c}$$

with

$$\mu^2 = 2\pi^2\epsilon/mC^2 = 2\pi^2\gamma/MC^2. \tag{3.2d}$$

The parameters X_j ($j = 1, 2$) are the soliton coordinates in the discrete lattice (X_1 being the position of the soliton in the first chain and X_2 that for the second chain). In the continuum limit of a homogeneous and uncoupled system, X_j are proportional to time τ, but in the discrete limit, they possess a rich and complex dependence on time (see the quoted paper on the discreteness effects). Substituting (3.1) and the *ansatz* (3.2) in the Lagrangian (2.1), we use the Lagrangian formalism to derive the following set of equations for the coordinates X_j of the centres of the coupled solitons:

$$\left(M_1 + 4A_1\theta_1^2\frac{\Delta m}{m}\operatorname{sech}^2\theta_1(1-X_1)\right)\ddot{X}_1 + \frac{1}{2}\left(\frac{\mathrm{d}M_1}{\mathrm{d}X_1}\right)\dot{X}_1^2 = A_1E_1\sin(2\pi X_1)$$
$$- \beta_1\sigma_1\sigma_2 d[C_{10} + C_{11}\cos(2\pi X_1)] - \alpha d\left(8\theta_1 + \frac{16\pi^2}{\sinh(\pi^2/\theta_1)}\cos(2\pi X_1)\right)$$
$$-4A_1\theta_1^3\left(\frac{\Delta m}{m}\dot{X}_1^2 + \frac{\Delta k}{k} + \frac{\Delta\epsilon}{\epsilon}\right)\operatorname{sech}^2\theta_1(1-X_1)\tanh\theta_1(1-X_1) - \frac{\mathrm{d}U_{12\mathrm{inh}}}{\mathrm{d}X_1} \tag{3.3a}$$

$$\left(M_2 + 4A_2\theta_2^2\frac{\Delta m}{m}\operatorname{sech}^2\theta_2(1-X_2)\right)\ddot{X}_2 + \frac{1}{2}\left(\frac{\mathrm{d}M_2}{\mathrm{d}X_2}\right)\dot{X}_2^2 = A_2E_2\sin(2\pi X_2)$$
$$+ \beta_1\sigma_1\sigma_2 d[C_{20} + C_{21}\cos(2\pi X_2)] - \alpha d\left(8\theta_2 + \frac{16\pi^2}{\sinh(\pi^2/\theta_2)}\cos(2\pi X_2)\right)$$
$$-4A_2\theta_2^3\left(\frac{\Delta M}{M}\dot{X}_2^2 + \frac{\Delta g}{g} + \frac{\Delta\gamma}{\gamma}\right)\operatorname{sech}^2\theta_2(1-X_2)\tanh\theta_2(1-X_2) - \frac{\mathrm{d}U_{12\mathrm{inh}}}{\mathrm{d}X_2} \tag{3.3b}$$

with

$$E_1 = (1-\beta_1\sigma_1\sigma_2)F_{11} - \beta_1\sigma_1\sigma_2\frac{16\pi^3}{\sinh(\pi^2/\theta_1)} \qquad F_{11} = \frac{\theta_1^3}{3}\frac{4\pi^3}{\sinh(\pi^2/\theta_1)}(29+l) \tag{3.4a}$$

$$C_{10} = 4\theta_1^3\left(\tfrac{4}{3} - \tfrac{7}{45}\theta_1^2\right) \tag{3.4b}$$

$$C_{11} = \frac{4\theta_1^2\pi^2}{\sinh(\pi^2/\theta_1)}\left[-\tfrac{16}{15}\theta_1^2(q+1)(q+4) + \tfrac{1}{3}(16+\tfrac{40}{3}\theta_1^2)(q+1) - (4+\tfrac{1}{3}\theta_1^2)\right] \tag{3.4c}$$

where

$$q = \pi^2/\mu^2.$$

Similar expressions can be written for E_2, C_{20} and C_{21} by replacing θ_1 by θ_2. For the derivation of E_j, C_{j0} and C_{j1}, one should consult Woafo *et al* (1992a). The quantities M_j are the masses of solitons in the coupled system. They can be expanded into Fourier series to give (see Woafo *et al* 1992a)

$$M_j \simeq M_{j0} + M_{j1}\cos(2\pi X_j) \tag{3.5a}$$

with

$$M_{j0} = 8A_j\theta_j \qquad M_{j1} = 16\pi^2 A_j/\sinh(\pi^2/\theta_j). \tag{3.5b}$$

The parameter d in (3.3), assumed small and constant, stands for the distance

$$d = X_1 - X_2 \tag{3.6}$$

between the centres of solitons of different chains. The interchain energy $U_{12\text{inh}}$ due to impurities is defined as

$$U_{12\text{inh}} = (a^2\theta_1 \Delta\beta/\pi^2)\,\text{sech}^2\,\theta_1(n - X_1) - d\ \text{sech}^2\,\theta_1(n - X_1)\tanh\theta_1(n - X_1) + 2\Delta\alpha \sin^2[d\theta_1\ \text{sech}\ \theta_1(n - X_1)]. \tag{3.7}$$

A similar expression can be obtained for X_2 by replacing X_1 by X_2 and θ_1 by θ_2. In this case, the minus sign before d is replaced by a plus sign.

Equations (3.3) and (3.7) show that the soliton of the first chain can be reflected or attracted (trapped) on impurity sites l and n while the soliton of the second chain can suffer similarly on sites p and n. The repulsive or attractive character of impurities depends on the soliton polarities and on the signs of the quantities

$$\Delta k/k + \Delta\epsilon/\epsilon \qquad \text{and} \qquad \Delta\beta$$

for the first chain and

$$\Delta g/g + \Delta\gamma/\gamma \qquad \text{and} \qquad \Delta\beta$$

for the second chain. This state of intrachain and interchain attractions and repulsions in different sites may create new effects on the dynamics of the coupled system, such as chaotic behaviour (Woafo and Kofané 1992). As shown below (section 4), the set of coupled and non-linear equations (3.3) has a rich variety of dynamical properties.

4. Analysis of the effective equations of motion for centres of coupled solitons

4.1. Motion of toplogical solitons in a homogeneous coupled system

When the chains of particles are homogeneous, the analysed model reduces to that considered by Woafo *et al* (1992a), but with some interesting changes and a new coupling. Indeed, in the present work, we have taken into account the lattice parameters, which differ from one chain to another ($m \neq M$, $k \neq g$, $\epsilon \neq \gamma$). This leads to different values for the soliton dynamical parameters in the two coupled chains (e.g. $E_1 \neq E_2$ and $M_1 \neq M_2$) and allows the discussion of section 4.3 since $U_{1\text{inh}}(0) \neq U_{2\text{inh}}(0)$. We have also considered the coupling between the nearest-neighbour particles of different chains. This adds the αd terms in (3.3). The coupling coefficient β modifies the PN barrier and consequently the frequency of vibrations for the pinned kink in the PN potential. If $\beta > 0$, there is a reduction (an increase) of the PN potential when the solitons have the same polarities (opposite polarities). The case $\beta < 0$ yields the inverse process.

4.2. Motion of solitons in inhomogeneous uncoupled chains

An interesting case is one where there is no coupling between the chains ($\beta = \alpha = 0$). Then, for each chain, there is an uncoupled equation of motion for the centre of the soliton. For instance, in the first chain, we have

$$\left(M_1 + 4A_1\theta_1^2\frac{\Delta m}{m}\operatorname{sech}^2\theta_1(l-X_1)\right)\ddot{X}_1 + \frac{1}{2}\left(\frac{dM_1}{dX_1}\right)\dot{X}_1^2 = A_1F_{11}\sin(2\pi X_1)$$
$$-4A_1\theta_1^3\left(\frac{\Delta m}{m}\dot{X}_1^2 + \frac{\Delta k}{k} + \frac{\Delta\epsilon}{\epsilon}\right)\operatorname{sech}^2\theta_1(l-X_1)\tanh\theta_1(l-X_1). \qquad (4.1a)$$

Taking $M_1 \simeq M_{10} = 8A_1\theta_1$, since the periodically varying part of M_1 can be neglected ($M_{11} \ll M_{10}$), the above equation takes the form

$$\left(1 + \frac{\theta_1^2}{2}\frac{\Delta m}{m}\operatorname{sech}^2\theta_1(l-X_1)\right)\ddot{X}_1 = \frac{\pi^3\theta_1}{6\sinh(\pi^2/\theta_1)}\left(\frac{2\pi^2}{\mu^2}+1\right)\sin(2\pi X_1)$$
$$-\frac{\mu^2}{2}\left(\frac{\Delta m}{m}\dot{X}_1^2 + \frac{\Delta k}{k} + \frac{\Delta\epsilon}{\epsilon}\right)\operatorname{sech}^2\theta_1(l-X_1)\tanh\theta_1(l-X_1). \qquad (4.1b)$$

With $\sigma_1 = 1$ and $l = 0$, one obtains

$$\left(1 + \frac{\mu^2}{2}\frac{\Delta m}{m}\operatorname{sech}^2\mu X_1\right)\ddot{X}_1 = \frac{\pi^2\mu}{6\sinh(\pi^2/\mu)}\left(\frac{2\pi^2}{\mu^2}+1\right)\sin(2\pi X_1)$$
$$+\frac{\mu^2}{2}\left(\frac{\Delta m}{m}\dot{X}_1^2 + \frac{\Delta k}{k} + \frac{\Delta\epsilon}{\epsilon}\right)\operatorname{sech}^2\mu X\tanh\mu X_1. \qquad (4.1c)$$

This last equation differs from (6.12) (with $\Gamma = F = 0$) of Braun and Kivshar (1991) on two points. The first difference is that, in our equation, the mass impurity renormalizes the acceleration coefficient. This renormalization comes naturally from the Lagrangian formalism. The second difference comes from the PN barrier $E_{PN} = F_{11}/\pi$ (the coefficient of $\sin(2\pi X_1)$), which in our work contains the supplementary multiplicative term $(2\pi^2/\mu^2+1)$. Its origin is in the order of truncation of the discreteness effects (see Woafo *et al* 1992a).

4.3. Case of single point impurity

This corresponds to the situation where all the impurities are localized in the same cell of the coupled lattice, for instance in cell 0 ($l = p = n = 0$). We assume for the analysis that follows that the distance between the solitons is equal to zero ($d = 0$). Then, we find that the soliton of the first chain moves in the effective potential

$$U_{\text{eff}} = U_{\text{PN}} + U_{1\text{inh}} \qquad (4.2a)$$

where

$$U_{\text{PN}} = U_0 + \tfrac{1}{2}A_1E_1\cos(2\pi X_1) \qquad (4.2b)$$

and

$$U_{1\text{inh}} = 2A_1\mu^2\left(\frac{\Delta k}{k} + \frac{\Delta\epsilon}{\epsilon} + \frac{a^2\Delta\beta}{2A_1\pi}\right)\operatorname{sech}^2\theta_1X_1 \qquad (4.2c)$$

where E_1 defined by equation (3.4*a*) depends on the coupling coefficient β, U_0 is an energy constant and $U_{j\text{inh}}$ is the potential energy due to inhomogeneous spring constants and substrate potential barriers. The influence of the α coupling disappears since $d = 0$.

If $\Delta m = 0$ and $M_1 \simeq M_{10}$, the soliton dynamics is then described by the following energy conservative equation

$$\tfrac{1}{2}M_{10}\dot{X}_1^2 + U_{\text{eff}} = E_k \tag{4.3}$$

where E_k is the total and constant energy of the soliton.

It must be noted that a similar equation can be written for the soliton of mass M_2, which propagates in the second chain. As shown in (4.2*a*), the effective potential contains two parts. The first part, U_{PN}, due to the discrete nature of the lattice, is the periodic PN potential. Its effects on soliton motion have already been analysed in section 4.1. The second part of the potential, $U_{1\text{inh}}$, is the impurity potential. It depends on Δk, $\Delta\epsilon$ and the variation $\Delta\beta$ of the coupling coefficient between the chains. $U_{1\text{inh}}$ has a sech form and is therefore localized around the impurity site; that is around $X_1 = 0$, where it has its extremum

$$U_{1\text{inh}}(0) = 2A_1\mu^2\left(\frac{\Delta k}{k} + \frac{\Delta\epsilon}{\epsilon} + \frac{a^2\Delta\beta}{2A_1\pi}\right). \tag{4.4}$$

The repulsive or the attractive character of the impurity potential depends on the sign of the quantity

$$\frac{\Delta k}{k} + \frac{\Delta\epsilon}{\epsilon} + \frac{a^2\Delta\beta}{2A_1\pi}$$

for the first chain or

$$\frac{\Delta g}{g} + \frac{\Delta\gamma}{\gamma} + \frac{a^2\Delta}{2A_2\pi}$$

for the second chain.

If it is positive (negative), the impurity potential is repulsive (attractive). This yields a possible competition between the impurities related to the elastic constants, the substrate potential barriers and the coupling coefficients between the chains of particles. We mention that when the solitons are far from the impurity site, their motion is modulated by the PN potential since $U_{j\text{inh}}$ goes to zero.

When the solition width is large enough ($\mu \ll 1$), the PN potential disappears and the soliton dynamics depends only on their interaction with the localized impurities. In this case, near the impurity site, the motion of the coupled solitons is described by the following set of ordinary differential equations:

$$\ddot{X}_1 = \tfrac{1}{2}\sigma_1\mu^3\left(\frac{\Delta k}{k} + \frac{\Delta\epsilon}{\epsilon} + \frac{a^2\Delta\beta}{2A_1\pi}\right)X_1 \tag{4.5a}$$

$$\ddot{X}_2 = \tfrac{1}{2}\sigma_2\mu^3\left(\frac{\Delta g}{g} + \frac{\Delta\gamma}{\gamma} + \frac{a^2\Delta\beta}{2A_2\pi}\right)X_2. \tag{4.5b}$$

We recall that $\sigma_j = 1$ for kinks and $\sigma_j = -1$ for antikinks. From (4.5), we can write the following results for the low-velocity solitons concerning the attractive and the repulsive character of the impurity:

First result. If

$$K_1 = \frac{\Delta k}{k} + \frac{\Delta \epsilon}{\epsilon} + \frac{a^2 \Delta \beta}{2A_1 \pi} > 0 \qquad \text{and} \qquad K_2 = \frac{\Delta g}{g} + \frac{\Delta \gamma}{\gamma} + \frac{a^2 \Delta \beta}{2A_2 \pi} > 0$$

(i) two coupled kinks are reflected by the impurity potential; (ii) two coupled antikinks are trapped; (iii) a kink of the first chain is reflected while an antikink of the second chain is trapped and vice versa.

Second result. If

$$K_1 < 0 \qquad \text{and} \qquad K_2 < 0$$

(i) two coupled kinks are trapped; (ii) two coupled antikinks are reflected; (iii) an antikink of the first chain is reflected while a kink of the second chain is trapped and vice versa.

Third result. If

$$K_1 > 0 \qquad \text{and} \qquad K_2 < 0$$

(i) a kink of the first chain is reflected while a kink of the second chain is trapped; (ii) an antikink of the first chain is trapped while a kink of the second chain is also trapped; (iii) a kink of the first chain is reflected while an antikink of the second chain is also reflected and vice versa.

Fourth result. This is obtained for

$$K_1 < 0 \qquad \text{and} \qquad K_2 > 0.$$

We then replace the first chain by the second one and vice versa in (i), (ii) and (iii) of the third result.

These interesting results show that the kink and the antikink do not see an impurity potential in the same manner. When a kink is reflected by an impurity potential, an antikink is attracted and vice versa. In the case of a repulsive impurity potential for solitons of both chains, the low-velocity coupled solitons are reflected. But with a sufficient kinetic energy greater than $|U_{1\mathrm{inh}}(0)| + |U_{2\mathrm{inh}}(0)|$, the coupled solitons will surmount the hills created by the impurity with more or less alteration of their topological structure and dynamical behaviour.

When the impurity potential is attractive for solitons of both chains, the low-velocity coupled solitons are trapped in the holes created by the impurity. Then, they oscillate with frequencies Ω_1 and Ω_2 defined as

$$\Omega_1^2 = -\tfrac{1}{2}\sigma_1 \mu^3 \left(\frac{\Delta k}{k} + \frac{\Delta \epsilon}{\epsilon} + \frac{a^2 \Delta \beta}{2A_1 \pi} \right) \tag{4.6a}$$

for the soliton of the first chain and

$$\Omega_2^2 = -\tfrac{1}{2}\sigma_2 \mu^3 \left(\frac{\Delta g}{g} + \frac{\Delta \gamma}{\gamma} + \frac{a^2 \Delta \beta}{2A_2 \pi} \right) \tag{4.6b}$$

for the soliton of the second chain. Since in general $A_1 \neq A_2$, $k \neq g$ and $\epsilon \neq \gamma$, we have $\Omega_1 \neq \Omega_2$. However, because of the radiative effects due to interactions between the solitons and the impurities, the soliton velocities go to zero with time (see Kivshar and Malomed 1989).

From the above analysis another interesting case appears where a soliton of one chain is reflected by the hill created by the impurities while that of the other chain is trapped in the hole also created by the impurities.

When the solition energy is so low that it is trapped by the PN barrier in the equilibrium site, $X_{jh} = h + \frac{1}{2}$ (h being an integer), that is midway between two particles of the lattice, then it oscillates around this site with frequency

$$\omega_{k1}^2 = \omega_{PN}^2 - \tfrac{1}{2}\sigma_1\mu^3 \left(\frac{\Delta k}{k} + \frac{\Delta\epsilon}{\epsilon} + \frac{a^2\Delta\beta}{2A_1\pi} \right) \frac{1 - 2\sinh^2\theta_1 X_{1h}}{\cosh^4\theta_1 X_{1h}} \tag{4.7}$$

where

$$\omega_{PN}^2 = 2\pi E_1/8\mu \tag{4.8}$$

for the soliton of the first chain. An identical equation can be written for the second chain. The PN frequency ω_{PN} decreases exponentially when the soliton width increases. It depends on the coupling coefficient β and on the relative polarity σ of the solitons that propagate along the coupled chains (Woafo *et al* 1992a).

4.4. *Case of mass impurities*

When the mass impurities Δm and ΔM are considered alone, the soliton centre position X_1 satisfies, in the case of negligible discreteness effects and for $d = 0$ ($l = p = n = 0$), the following equation:

$$\left(M_1 + 4A_1\theta_1^2 \frac{\Delta m}{m} \operatorname{sech}^2\theta_1 X_1 \right) \ddot{X}_1 = 4A_1\theta_1^3 \frac{\Delta m}{m} \dot{X}_1^2 \operatorname{sech}^2\theta_1 X_1 \tanh\theta_1 X_1 \tag{4.9}$$

which in the phase space $(X_1, \dot{X}_1)$ takes the form

$$\dot{X}_1 = \left(\frac{M_{10}}{M_{10} + 4A_1\theta_1^2(\Delta m/m)\operatorname{sech}^2\theta_1 X_1} \right)^{1/2} \dot{X}_0 \tag{4.10}$$

where $\dot{X}_0$ is the soliton initial velocity. A similar equation can be written for X_2. At the impurity site $X_1 = 0$, the soliton velocity takes the value

$$\dot{X}_{1\text{site}} = \left(\frac{M_{10}}{M_{10} + 4A_1\theta_1^2(\Delta m/m)} \right)^{1/2} \dot{X}_0. \tag{4.11}$$

It therefore appears (in connection with the sech function shape) that, in the case of a heavy mass impurity ($\Delta m > 0$), the soliton velocity, initially constant, decreases when approaching the impurity. Just after, the velocity increases until it reaches the initial constant value. The inverse process occurs in the case of a light impurity ($\Delta m < 0$). When discreteness effects are associated with the mass impurity, it can be shown that $\dot{X}_1$ and X_1 are linked by the equation (obtained through the method of variations of constants)

$$\dot{X}_1^2 = \frac{C - (A_1E_1/\pi)\cos(2\pi X_1)}{M_{10} + 4A_1\theta_1^2(\Delta m/m)\operatorname{sech}^2\theta_1 X_1} \tag{4.12}$$

where C is an arbitrary constant depending, as shown below, on the soliton initial velocity. According to its value, the kink motion is separated into two modes: namely the transmission mode for which the soliton goes through the impurity zone and the reflection mode. Indeed, assuming that the kink velocity vanishes at the impurity site, that is $\dot{X}_1 = 0$ for $X_1 = 0$, a critical value C_{cr} for C is defined as

$$C_{cr} = A_1 E_1/\pi. \tag{4.13}$$

For $C > C_{cr}$, the kink is transmitted through the impurity, while for $C < C_{cr}$, there is a reflection. Similar qualitative results have been obtained by Braun and Kivshar (1991) and Mefougue *et al* (1992). To derive the threshold velocity that separates the two propagation modes, we assume that the kink is initially pinned in a PN well, far from the impurity site, where it has a minimum energy. The assumption leads to

$$C = M_{10}\dot{X}_0^2 - A_1 E_1/\pi. \tag{4.14}$$

From (4.13) and (4.14), the threshold velocity $\dot{X}_{0cr}$ is defined as

$$\dot{X}_{0cr}^2 = 2A_1 E_1/\pi M_{10}. \tag{4.15}$$

It is just the critical velocity under which a kink is trapped by the PN barrier. Taking into account the variations of k and ϵ, one can show that the threshold velocity is

$$\dot{X}_{0cr}^2 = \frac{2A_1 E_1}{\pi M_{10}} + \frac{M_{10} + 4A_1\theta_1^2 \Delta m/m}{M_{10}} \left(\frac{\Delta k}{k} + \frac{\Delta\epsilon}{\epsilon} \right). \tag{4.16}$$

This last equation shows that the critical velocity is related to the local variations of mass, of the elastic constant k and of the substrate potential barrier ϵ. We recall that the above analysis can also be done in the second chain (for X_2) and the effects of the coupling appear on the relative polarity σ and the coupling coefficient β through E_j.

5. Conclusion

In this work, we have considered the motion of topological solitons in weakly coupled sine–Gordon chains with local impurities. The coupling takes into account the interaction between the relative displacements of particles in both chains and the interaction between a particle of one chain with its nearest neighbour of the other chain. Various types of impurity localized in different sites have been considered. They correspond to local variations of masses of particles, of elastic constants, of substrate potential barriers and of the coupling coefficients between the chains. In the discrete limit, we have used an *ansatz* based on the well known exact topological soliton of the one-dimensional SG equation. The Lagrangian formalism has been used to show that the dynamics of the coupled solitons can be described by a set of two coupled non-linear differential equations whose coefficients and potential forces depend on the different locations of impurities and on the coupling coefficients between the chains of particles. The attraction (or trapping) and reflection sites of a soliton of the first chain are different from those of a soliton of the second chain. The repulsive or attractive character of impurities depends on the soliton polarities and on the signs of different impurity potential barriers. By considering a point impurity, it has been shown that the coupled topological solitons with low velocities (kink–kink, kink–antikink and antikink–antikink pairs) can be reflected or attracted. The threshold conditions for soliton reflection by the impurities has been derived. A situation has also appeared where the soliton of one chain is reflected while that of the other chain is trapped.

The general case where the impurity sites l, p and n are different and the case of non-zero value for d may yield interesting new effects. The high- and low-velocity regimes should be analysed and particular attention should be given to the situation where one of the coupled solitons encounters a hill while the other encounters a hole (the hole and hill potentials created by the impurities). The radiative effects have not yet been considered in the present work. However, in a model such as ours, there are various sources of radiative effects: emission caused by the discreteness effects (Ishimori and Munakatta 1982, Peyrard and Kruskal 1984, Boesch *et al* 1989), emission due to the interaction of solitons with impurities (Kivshar and Malomed 1989) and excitation of localized oscillations around the impurities by the topological solitons (Fraggis *et al* 1989). The details of each of these phenomena are under investigation.

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References

Boesch R, Willis C R and El-batanouny M 1989 *Phys. Rev.* B **40** 2284
Braun O M and Kivshar Y S 1990 *Phys. Lett.* **149A** 119
—— 1991 *Phys. Rev.* B **43** 1060
Braun O M, Kivshar Y S and Kosevich A M 1988 *J. Phys. C: Solid State Phys.* **21** 3881
Combs J A and Yip S 1983 *Phys. Rev.* B **28** 6873
—— 1984 *Phys. Rev.* B **29** 438
Coutinho S, Pitanga P and Lederer P 1981 *Phys. Rev.* B **23** 4567
Dikandé A M and Kofané T C 1992 *Phys. Scr.* submitted
Fraggis T, Pnevmatikos St and Economou E N 1989 *Phys. Lett.* **142A** 361
Ishimori Y and Munakatta T 1982 *J. Phys. Soc. Japan* **51** 3367
Kivshar Y S and Malomed B A 1988 *Phys. Rev.* B **37** 9325
—— 1989 *Rev. Mod. Phys.* **61** 763
Kunz C and Combs J A 1985 *Phys. Rev.* B **37** 527
McLaughlin D W and Scott A C 1978 *Phys. Rev.* A **18** 1652
Mefougue R, Woafo P and Kofané T C 1992 *Physica* B submitted
Peyrard M and Kruskal D 1984 *Physica* D **14** 88
Peyrard M and Remoissenet M 1982 *Phys. Rev.* B **26** 2886
Pouget J, Aubry S, Bishop A R and Lomdahl P S 1989 *Phys. Rev.* B **39** 9500
Trullinger S E and Sasaki K 1987 *Physica* D **28** 181
Willis C R and Boesch R 1990 *Phys. Rev.* B **41** 4570
Woafo P and Kofané T C 1992 *Phys. Scr.* at press
Woafo P, Kofané T C and Bokosah A S 1991a *J. Phys.: Condens. Matter* **3** 2279
—— 1991b *Phys. Lett.* **160A** 237
—— 1992a *J. Phys.: Condens. Matter* **4** 809
—— 1992b *J. Phys.: Condens. Matter* **4** 3389
Zhang Chun-Ting 1987 *Phys. Rev.* A **35** 886
Zhang Fei, Kivshar Y S and Vasquez L 1992 *Phys. Rev.* A **45** 6019